

## Continuum states in generalized Swanson models

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## FAST TRACK COMMUNICATION

# Continuum states in generalized Swanson models

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Online at [stacks.iop.org/JPhysA/42/052002](http://stacks.iop.org/JPhysA/42/052002)**Abstract**

A one-to-one correspondence is known to exist between the spectra of the discrete states of the non-Hermitian Swanson-type Hamiltonian  $H = A^\dagger A + \alpha A^2 + \beta A^{\dagger 2}$  ( $\alpha \neq \beta$ ) and an equivalent Hermitian Schrödinger Hamiltonian  $h$ , the two Hamiltonians being related through a similarity transformation. In this work, we consider the continuum states of  $h$ , and examine the nature of the corresponding states of  $H$ .

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The spectrum of the non-Hermitian Swanson Hamiltonian,  $H = a^\dagger a + \alpha a^2 + \beta a^{\dagger 2}$  ( $\alpha \neq \beta$ ), has been found to have a one-to-one correspondence with that of the conventional (Hermitian) Harmonic oscillator Hamiltonian  $h$  [1]. Subsequent works [2–4] have shown that by replacing the Harmonic oscillator annihilation and creation operators  $a$  and  $a^\dagger$  by the generalized annihilation and creation operators  $A$  and  $A^\dagger$  respectively, the Swanson Hamiltonian could be generalized to include other interactions, and thus have the generalized form  $H = A^\dagger A + \alpha A^2 + \beta A^{\dagger 2}$ , ( $\alpha \neq \beta$ ). In these cases too,  $H$  can be mapped to an equivalent Hermitian Hamiltonian  $h$ , with the help of a similarity transformation (say  $\rho$ ), and the bound state energies were found to be the same for both the Hamiltonians (i.e.,  $H$  and  $h$ ). However, when  $h$  has both discrete and continuous spectra, no correspondence has been established to date between the continuum states of  $h$  and those (if any) of  $H$ . It may be mentioned here that two works dealing with one-dimensional scattering in non-Hermitian quantum mechanics (and hence, dealing with states having continuous spectra) deserve special mention in this regard. The first of these is the review article on complex absorbing potentials [5], dealing with one-dimensional scattering in non-Hermitian quantum mechanics, with particular emphasis on complex  $\mathcal{PT}$ -symmetric potentials. The second work gives a more detailed study of one-dimensional scattering in  $\mathcal{PT}$ -symmetric potentials, in particular, with some explicit examples of solvable potentials [6, 7].

However, in this work we shall not investigate scattering in generalized Swanson models. Rather, our aim is to focus our attention on those states of  $h$  having continuous energies, and explore the nature of the corresponding states of  $H$ . We shall also find the probability current

density and charge density for the  $\eta$ -pseudo Hermitian generalized Swanson Hamiltonian  $H$ . Our studies will be based on two well-known interactions, namely, the Pöschl–Teller and Morse models.

It is well known by now that a quantum system described by a  $\eta$ -pseudo Hermitian Hamiltonian  $H$ , can be mapped to an equivalent system described by its corresponding Hermitian counterpart  $h$ , with the help of a similarity transformation  $\rho$  [3, 8, 9],

$$h = \rho H \rho^{-1}. \tag{1}$$

So we start with the generalized Swanson Hamiltonian

$$H = A^\dagger A + \alpha A^2 + \beta A^{\dagger 2}, \quad \alpha \neq \beta \tag{2}$$

with solutions  $\psi(x)$  satisfying the eigenvalue equation

$$H\psi(x) = E\psi(x),$$

where  $\alpha$  and  $\beta$  are real, dimensionless constants. Since we are dealing with non-Hermitian  $H$ ,  $\alpha \neq \beta$ . Here  $A$  and  $A^\dagger$  are generalized annihilation and creation operators given by

$$A = \frac{1}{\sqrt{1-\alpha-\beta}} \left\{ \frac{d}{dx} + W(x) \right\}, \quad A^\dagger = \frac{1}{\sqrt{1-\alpha-\beta}} \left\{ -\frac{d}{dx} + W(x) \right\}. \tag{3}$$

If we apply to (2) a transformation of the form [10]

$$\psi(x) = \rho^{-1} \phi(x) \tag{4}$$

where

$$\rho = e^{-\mu \int W(x) dx}, \quad \text{with} \quad \mu = \frac{\alpha - \beta}{1 - \alpha - \beta}, \quad \alpha + \beta \neq 1 \tag{5}$$

(2) reduces to the following Hermitian Schrödinger-type Hamiltonian, with the same eigenvalue  $E$  (in units  $\hbar = 2m = 1$ )

$$h\phi(x) = \left( -\frac{d^2}{dx^2} + V(x) \right) \phi(x) = E\phi(x). \tag{6}$$

It may be recalled that the parameters  $\alpha, \beta$  must obey certain constraints, namely, [3, 4]

$$\alpha + \beta < 1, \quad 4\alpha\beta < 1. \tag{7}$$

Writing  $V(x)$  in (6) in the supersymmetric form ( $w^2 - w'$ ) [11, 12],  $W(x)$  and  $w(x)$  are found to be inter-related through

$$V(x) = w^2(x) - w'(x) = \left( \frac{\sqrt{1-4\alpha\beta}}{1-\alpha-\beta} W(x) \right)^2 - \frac{1}{(1-\alpha-\beta)} W'(x). \tag{8}$$

Since we are dealing with  $\eta$ -pseudo Hermitian Hamiltonians,  $H$  obeys the relationship [13]

$$H^\dagger = \eta H \eta^{-1} \quad \text{or} \quad H^\dagger \eta = \eta H \tag{9}$$

where  $\rho$  is the positive square root of  $\eta$  ( $\rho = \sqrt{\eta}$ ) [3, 8].

So, whereas  $\phi$  should obey the conventional normalization conditions,  $\psi$  should follow  $\eta$  inner product and hence be normalized according to  $\langle \psi_m | \eta | \psi_n \rangle = \delta_{m,n}$  [13].

Now let us look at the continuity condition for  $\phi$ , namely,

$$\frac{\partial}{\partial t} \phi^* \phi + \nabla \cdot j = 0 \tag{10}$$

where  $\phi^* \phi$  represents the conventional charge density and  $j$  is the conventional current density, given by

$$j = i \left( \frac{\partial \phi^*}{\partial x} \phi - \phi^* \frac{\partial \phi}{\partial x} \right) \tag{11}$$

in one dimension.

If we apply the inverse transformation of (4) to  $\phi$ , then the equivalent continuity equation for  $\psi$  in the pseudo Hermitian picture can be cast in the form

$$\frac{\partial}{\partial t} \chi + \nabla \cdot \bar{j} = 0 \tag{12}$$

provided we identify  $\chi$  and  $\bar{j}$  with

$$\chi = \psi^* \eta \psi, \quad \bar{j} = i\eta \left( \frac{\partial \psi^*}{\partial x} \psi - \psi^* \frac{\partial \psi}{\partial x} \right). \tag{13}$$

Thus  $\chi$  plays the role of charge density and  $\bar{j}$  that of current density for the generalized, pseudo Hermitian Swanson Hamiltonian. The states of  $H$  with discrete energies have been found to have a one-to-one correspondence with similar states of  $h$ . Our aim in this work is to see what happens to those states of the Hermitian Hamiltonian  $h$  with continuous spectra when we go over to the corresponding non-Hermitian Swanson Hamiltonian  $H$ , with the help of the similarity transformation  $h = \rho H \rho^{-1}$ . In particular, we shall do so for a couple of explicit examples, namely, the Pöschl–Teller and the Morse models.

**Pöschl–Teller interaction**

Following [4], to map the non-Hermitian Hamiltonian  $H$  to its Hermitian Schrödinger equivalent  $h$ , one needs to solve the highly non-trivial Riccati equation (8) analytically. This is possible only if we take  $W(x)$  and  $w(x)$  to be of the same form. For the Pöschl–Teller model taking

$$W(x) = \lambda_2 \sigma \tanh \sigma x, \quad w(x) = \lambda_1 \sigma \tanh \sigma x \quad \lambda_{1,2} > 0 \tag{14}$$

the potential term in the Schrödinger-type Hermitian Hamiltonian  $h$  is obtained as

$$V(x) = \lambda_1^2 \sigma^2 - \lambda_1 (\lambda_1 + 1) \sigma^2 \operatorname{sech}^2 \sigma x = \lambda_2^2 \sigma^2 \frac{1 - 4\alpha\beta}{(1 - \alpha - \beta)^2} - \zeta \sigma^2 \operatorname{sech}^2 \sigma x \tag{15}$$

solving which gives the unknown parameter  $\lambda_1$  in terms of the known one  $\lambda_2$ ,

$$\lambda_1 = \frac{\sqrt{1 + 4\zeta} - 1}{2} \quad \text{where} \quad \zeta = \frac{\lambda_2^2(1 - 4\alpha\beta) + \lambda_2(1 - \alpha - \beta)}{(1 - \alpha - \beta)^2}. \tag{16}$$

Introducing a new variable

$$y = \cosh^2 \sigma x \tag{17}$$

and writing the solutions of  $h$  as

$$\phi = y^{\frac{\lambda_1 + 1}{2}} u(y) \tag{18}$$

reduces equation (6) to the hypergeometric equation

$$y(1 - y)u'' + \left\{ \left( \lambda_1 + \frac{3}{2} \right) - (\lambda_1 + 2)y \right\} u' - \frac{1}{4} \left\{ (\lambda_1 + 1)^2 - \frac{\epsilon}{\sigma^2} \right\} u = 0 \tag{19}$$

with the complete solution

$$u = A_1 {}_2F_1 \left( a, b, \frac{1}{2}; 1 - y \right) + A_2 (1 - y)^{1/2} {}_2F_1 \left( a + \frac{1}{2}, b + \frac{1}{2}, \frac{3}{2}; 1 - y \right) \tag{20}$$

where

$$\epsilon = E - \lambda_1^2 \sigma^2 \tag{21}$$

and  $a, b$  are given below. Therefore, for  $\epsilon < 0$ , say  $\epsilon = -\kappa^2\sigma^2$ , the complete solution of  $h$  is obtained as [14]

$$\phi = A_1 y^{\frac{1}{2}(\lambda_1+1)} {}_2F_1\left(a, b, \frac{1}{2}; 1-y\right) + A_2 y^{\frac{1}{2}(\lambda_1+1)} (1-y)^{1/2} {}_2F_1\left(a + \frac{1}{2}, b + \frac{1}{2}, \frac{3}{2}; 1-y\right) \quad (22)$$

with

$$a = \frac{1}{2} \left( \lambda_1 + 1 - \frac{\kappa}{\sigma} \right), \quad b = \frac{1}{2} \left( \lambda_1 + 1 + \frac{\kappa}{\sigma} \right). \quad (23)$$

Thus for negative energies, bound state normalizable solutions exist only if either  $a = -n$  or  $a + \frac{1}{2} = -n$  giving

$$E_n = -(\lambda_1 - n)^2\sigma^2, \quad n = 0, 1, \dots, \leq \lambda_1. \quad (24)$$

Consequently, the bound state energies  $E$  for the generalized Swanson Hamiltonian  $H$ , for this particular model, are also given by (24). Since the form of  $W(x)$  in (14) yields

$$\rho^{-1} = (\cosh \sigma x)^{\lambda_2\mu} \quad (25)$$

the bound state solutions of  $H$ , obtained from (22) by applying (4), are found to be

$$\begin{aligned} \psi(x) = & A_1 \cosh^{\lambda_2\mu+\lambda_1+1} \sigma x {}_2F_1\left(a, b, \frac{1}{2}; -\sinh^2 \sigma x\right) \\ & + A_2 \cosh^{\lambda_2\mu+\lambda_1+1} \sigma x \sinh \sigma x {}_2F_1\left(a + \frac{1}{2}, b + \frac{1}{2}, \frac{3}{2}; -\sinh^2 \sigma x\right) \end{aligned} \quad (26)$$

with either  $A_1 = 0$  or  $A_2 = 0$ , for well-behaved, normalizable solutions; i.e., the eigenstates are either even ( $A_2 = 0$ ) or odd ( $A_1 = 0$ ). However, in this work, our interest lies in those states of  $H$  which correspond to the continuum states (i.e., positive energy states, with  $\epsilon = k^2\sigma^2$ ) of  $h$ , rather than the bound states. If  $\phi(x)$  be the continuum states of  $h$ , with [7, 14, 16],

$$\phi(x) = a_1\phi_1(x) + a_2\phi_2(x) \quad (27)$$

then  $\phi_1$  and  $\phi_2$  are of the form

$$\phi_1 = (\cosh \sigma x)^{\lambda_1+1} {}_2F_1\left(a, b, \frac{1}{2}; -\sinh^2 \sigma x\right) \quad (28)$$

$$\phi_2 = (\cosh \sigma x)^{\lambda_1+1} \sinh \sigma x {}_2F_1\left(a + \frac{1}{2}, b + \frac{1}{2}, \frac{3}{2}; -\sinh^2 \sigma x\right) \quad (29)$$

where

$$a = \frac{1}{2} \left( \lambda_1 + 1 + i\frac{k}{\sigma} \right), \quad b = \frac{1}{2} \left( \lambda_1 + 1 - i\frac{k}{\sigma} \right). \quad (30)$$

The continuum states of  $H$  are obtained (by applying equation (4) to  $\phi(x)$ ) in the same form as (26), with  $a, b$  as defined in (30). Using the asymptotic limit of the hypergeometric functions  ${}_2F_1(a, b, c; z)$ , for large  $|z|$  [15], namely,

$$\begin{aligned} {}_2F_1(a, b, c; z) \sim & \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)} (-z)^{-a} {}_2F_1\left(a, 1-c+a, 1-b+a; \frac{1}{z}\right) \\ & + \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)} (-z)^{-b} {}_2F_1\left(b, 1-c+b, 1-a+b; \frac{1}{z}\right) \end{aligned} \quad (31)$$

along with the fact that when  $x \rightarrow \pm\infty$ ,  $-\sinh^2 \sigma x \rightarrow -2^{-2} e^{2\sigma|x|}$ , and  $\cosh^2 \sigma x \rightarrow 2^{-2} e^{2\sigma|x|}$ , it can be shown by simple straightforward algebra that the solutions  $\phi$  have asymptotic behaviour of the form [14]

$$\phi(x) = \begin{cases} e^{ikx} + R e^{-ikx}, & \text{for } x < 0 \\ T e^{ikx}, & \text{for } x > 0 \end{cases} \quad (32)$$

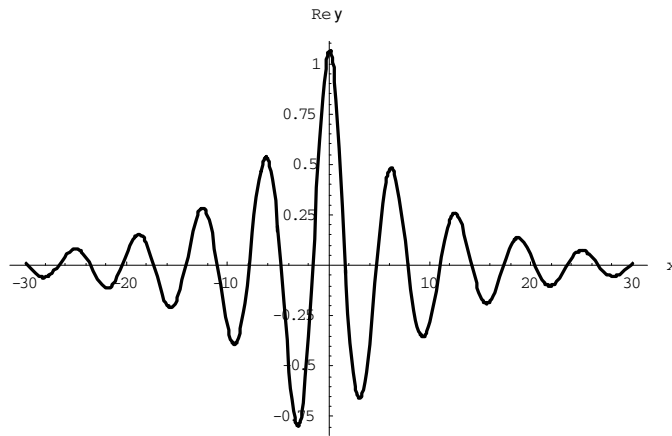


Figure 1. A plot of  $\text{Re } \psi$  versus  $x$  for the damped wave solutions with continuous spectra, for  $-\text{ve } \mu$ .

provided the coefficients  $a_1, a_2$  obey certain restrictions, giving the reflection and transmission amplitudes in (32) by

$$R = \frac{1}{2}(e^{2i\varphi_e} + e^{2i\varphi_o}), \quad T = \frac{1}{2}(e^{2i\varphi_e} - e^{2i\varphi_o}) \tag{33}$$

with

$$\varphi_e = \arg \frac{\Gamma(ik/\alpha) e^{-i\frac{k}{\alpha} \log 2}}{\Gamma(\frac{\lambda+1}{2} + i\frac{k}{2\alpha}) \Gamma(\frac{-\lambda}{2} + i\frac{k}{2\alpha})} \tag{34}$$

$$\varphi_o = \arg \frac{\Gamma(ik/\alpha) e^{-i\frac{k}{\alpha} \log 2}}{\Gamma(\frac{\lambda}{2} + i\frac{k}{2\alpha}) \Gamma(\frac{1-\lambda}{2} + i\frac{k}{2\alpha})}. \tag{35}$$

Thus the conventional conservation law  $|R|^2 + |T|^2 = 1$  is obeyed.

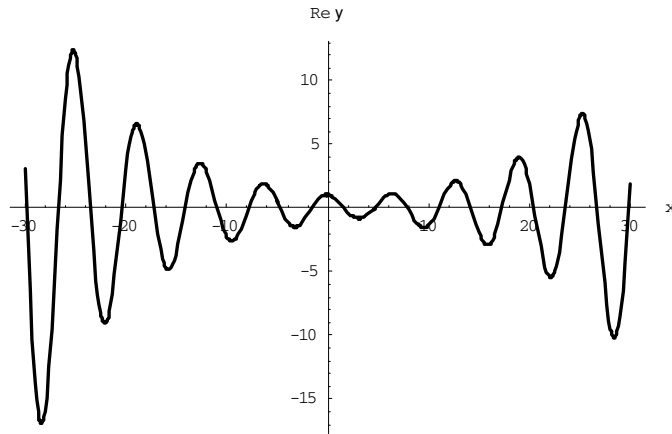
We shall now use the relationship  $\psi(x) = \rho^{-1}\phi(x)$ , to obtain those states of  $H$  which correspond to the continuum states of  $h$ , and hence possess continuous energies. If these solutions are written as

$$\psi(x) = B_1\psi_1(x) + B_2\psi_2(x) \tag{36}$$

then, applying (4) to (32) yields the following asymptotic behaviour of  $\psi$ :

$$\psi(x) \sim \begin{cases} e^{\lambda_2\mu|x|} \{e^{ikx} + R e^{-ikx}\}, & \text{for } x < 0 \\ e^{\lambda_2\mu|x|} T e^{ikx}, & \text{for } x > 0 \end{cases} \tag{37}$$

with  $R$  and  $T$  as defined in (33). However, unlike in the Hermitian case, here  $R$  and  $T$  can no longer be identified as the reflection and transmission amplitudes. Hence the plane-wave solutions of the Hermitian Hamiltonian  $h$  are replaced in the case of the pseudo Hermitian Swanson-type Hamiltonian  $H$ , by progressive waves for  $\mu > 0$ , i.e.,  $\alpha > \beta$ , or damped waves for  $\mu < 0$ , i.e.,  $\alpha < \beta$ . Thus the role played by  $\mu$ , i.e. the parameters  $\alpha, \beta$  are quite significant in the case of the continuum states. However, for the bound states no significant change is introduced in the nature of the solutions of  $h$  and  $H$ , depending on the sign of  $\mu$ . For instance, for  $\alpha = \frac{1}{8}, \beta = \frac{1}{4}, \lambda_2 = 5$ , we obtain  $\zeta = 384, \mu = -\frac{1}{5}, \lambda_1 = 9.31$ , implying that there are ten bound states ( $n = 0, 1, 2, \dots, 9$ ) in both  $h$  and  $H$  [3]. The real part of such a damped wave solution for  $\psi$  as given in (37) is plotted in figure 1,



**Figure 2.** A plot of  $\text{Re } \psi$  versus  $x$  for the progressive wave solutions with continuous spectra, for  $+\text{ve } \mu$ .

for the parameter values mentioned above, i.e.,  $\alpha = \frac{1}{8}, \beta = \frac{1}{4}, \lambda_2 = 5, \mu = -1/5$ , while figure 2 shows the real part of a progressive wave solution for the same  $\psi$  for the parameter values  $\alpha = \frac{1}{2}, \beta = \frac{1}{4}, \lambda_2 = 1, \mu = 1$ , so that  $\zeta = 12$  and  $\lambda_1 = 3$ .

**Morse interaction**

To check whether the results obtained so far are peculiar to the particular interaction studied, we consider a second example, namely, the Morse model. Taking the following ansatz for  $W(x)$  and  $w(x)$ :

$$W(x) = a_2\sigma - b_2\sigma e^{-\sigma x}, \quad w(x) = a_1\sigma - b_1\sigma e^{-\sigma x} \quad a_{1,2}, b_{1,2} > 0. \tag{38}$$

This yields the following expression for  $V(x)$  in equation (8):

$$\begin{aligned} V(x) &= a_1^2\sigma^2 + b_1^2\sigma^2 e^{-2\sigma x} - 2b_1 \left( a_1 + \frac{1}{2} \right) \sigma^2 e^{-\sigma x} \\ &= \frac{1 - 4\alpha\beta}{(1 - \alpha - \beta)^2} a_2^2\sigma^2 + \frac{1 - 4\alpha\beta}{(1 - \alpha - \beta)^2} b_2^2\sigma^2 e^{-2\sigma x} \\ &\quad - \frac{(1 - 4\alpha\beta)2a_2 + (1 - \alpha - \beta)}{(1 - \alpha - \beta)^2} b_2^2\sigma^2 e^{-\sigma x} \end{aligned} \tag{39}$$

with

$$a_1 = \frac{\sqrt{1 - 4\alpha\beta}}{1 - \alpha - \beta} a_2 + \frac{1}{2\sqrt{1 - 4\alpha\beta}} - \frac{1}{2}, \quad b_1 = \frac{\sqrt{1 - 4\alpha\beta}}{1 - \alpha - \beta} b_2. \tag{40}$$

Substituting

$$z = 2b_1 e^{-\sigma x}, \quad \phi = e^{-z/2} z^s u(z) \tag{41}$$

so that  $-\infty < x < \infty$  transforms over to  $0 \leq z < \infty$ , and equation (6) reduces to

$$u'' + \left( \frac{2s + 1}{z} - 1 \right) u' + \left( \frac{a_1 - s}{z} + \frac{\epsilon/\sigma^2 + s^2}{z^2} \right) u = 0 \tag{42}$$

where  $\epsilon = E - a_1^2 \sigma^2$ . For bound states,  $\epsilon = -\kappa^2$  giving  $s = \pm \kappa/\sigma$ . Thus the bound state solutions of the Hermitian Hamiltonian  $h$  and its corresponding non-Hermitian one  $H$  are respectively given by [11],

$$\phi_n = e^{-z/2} z^s L_n^{2s}(z) \tag{43}$$

$$\psi_n = \rho^{-1} \phi_n \sim e^{-(1+\mu b_2/b_1)z/2} z^{s-\mu a_2} L_n^{2s}(z) \tag{44}$$

where  $L_n^{2s}$  are associated Laguerre polynomials [15],  $s = a_1 - n$  and normalization requirement restricts  $s$  to positive values only. The bound state energies of both the Hermitian and non-Hermitian systems are obtained as

$$\epsilon_n = -(a_1 - n)^2 \sigma^2 \quad \text{giving} \quad E_n = (2a_1 - n)n\sigma^2, \quad n = 0, 1, 2, \dots < a_1. \tag{45}$$

However, our aim in this work is to explore those states of  $H$  whose Hermitian equivalents are the continuum states (i.e. positive energies) of  $h$ . We proceed in a manner analogous to the previous section, with the following form of  $\rho$  for this model:

$$\rho = \left(\frac{z}{2b_1}\right)^{\mu a_2} e^{-\frac{\mu b_2}{2b_1}z}, \quad \text{where} \quad \frac{\mu b_2}{2b_1} = \frac{\alpha - \beta}{2\sqrt{1 - 4\alpha\beta}}. \tag{46}$$

Writing  $\epsilon = E - a_2 \sigma^2 = k^2$ , so that  $s = \pm \frac{ik}{\sigma}$ , the continuum state solutions of  $h$  are found to be [17]

$$\phi = A_1 e^{-z/2} z^{ik/\sigma} F\left(\frac{ik}{\sigma} - a_1, \frac{2ik}{\sigma} + 1, z\right) + A_2 e^{-z/2} z^{-ik/\sigma} F\left(-\frac{ik}{\sigma} - a_1, -\frac{2ik}{\sigma} + 1, z\right) \tag{47}$$

where  $F(a, b, z)$  are the Kummer confluent hypergeometric functions [15]. Since the potential  $V(x)$  goes to infinity for large negative values of  $x$  (i.e.,  $z \rightarrow 0$ ), the wavefunction  $\phi(x)$  should vanish in this region :  $\phi(x \rightarrow -\infty) \rightarrow 0$ . Using the properties of confluent hypergeometric functions [15],

$$\phi(z \rightarrow 0) \rightarrow A_1 \frac{\Gamma\left(\frac{2ik}{\sigma} + 1\right)}{\Gamma\left(\frac{ik}{\sigma} - a_1\right)} + A_2 \frac{\Gamma\left(-\frac{2ik}{\sigma} + 1\right)}{\Gamma\left(-\frac{ik}{\sigma} - a_1\right)} \rightarrow 0 \tag{48}$$

giving the ratio  $A_2 : A_1$  as

$$\frac{A_2}{A_1} = -\frac{\Gamma\left(\frac{2ik}{\sigma} + 1\right) \Gamma\left(-\frac{ik}{\sigma} - a_1\right)}{\Gamma\left(-\frac{2ik}{\sigma} + 1\right) \Gamma\left(\frac{ik}{\sigma} - a_1\right)}. \tag{49}$$

Similarly for large values of  $x$  in the positive direction,

$$\phi(x \rightarrow \infty) \rightarrow A_1 (2b_1)^{ik/\sigma} e^{-ikx} + A_2 (2b_1)^{-ik/\sigma} e^{ikx}. \tag{50}$$

Thus  $\phi(x)$  can be written in the form (33), with

$$R = (2b_1)^{-ik/\sigma} \frac{\Gamma\left(\frac{2ik}{\sigma} + 1\right) \Gamma\left(-\frac{ik}{\sigma} - a_1\right)}{\Gamma\left(-\frac{2ik}{\sigma} + 1\right) \Gamma\left(\frac{ik}{\sigma} - a_1\right)} \tag{51}$$

and  $T = 0$ , which is expected as the potential goes to  $\infty$  at  $x \rightarrow -\infty$ . Instead of going into detailed calculations, we just quote the solutions of  $H$  having a one-to-one correspondence with those states of  $h$  having continuous spectra

$$\begin{aligned} \psi(x \rightarrow \infty) &\sim \left(\frac{z}{2b_1}\right)^{-\mu a_2} e^{\frac{\mu b_2}{2b_1}z} \phi \\ &= e^{\mu a_2 \sigma x} (c_1 e^{-ikx} + c_2 e^{ikx}). \end{aligned} \tag{52}$$



Once again, we obtain a result identical to that obtained in the previous case namely, depending on the sign of  $\mu$ , states with continuous spectra are either progressive waves ( $\mu > 0$ ) or damped waves ( $\mu < 0$ ) in the non-Hermitian generalized Swanson Hamiltonian.

To conclude, we have studied states with continuous spectra in a class of  $\eta$ -pseudo Hermitian Hamiltonians, which are of the generalized Swanson type, namely,  $H = \mathcal{A}^\dagger \mathcal{A} + \alpha \mathcal{A}^2 + \beta \mathcal{A}^{\dagger 2}$ ,  $\alpha \neq \beta$ , and hence can be mapped to an equivalent Hermitian Schrödinger-type Hamiltonian  $h$  with the help of a similarity transformation  $\rho$ . We have also obtained the modified continuity equation the solutions of such non-Hermitian Hamiltonians should obey, and obtained new definitions for the charge density and current density. In particular, we have analysed the positive energy solutions for two one-dimensional pseudo Hermitian models, whose Hermitian equivalents are the Pöschl–Teller and the Morse interactions. In both cases it is observed that the relative strength of the parameters  $\alpha$  and  $\beta$  plays a crucial role in determining the nature of the solutions with continuous spectra. If  $\alpha > \beta$  implying  $\mu > 0$ , then the continuum states of  $h$  represented by plane-wave solutions are replaced by progressive waves in the case of the corresponding non-Hermitian Hamiltonian  $H$ . Similarly, for  $\alpha < \beta$  implying  $\mu < 0$ , the continuum energy states of the generalized Swanson Hamiltonian  $H$  are represented by damped waves. This asymmetrical nature of the parameters  $\alpha, \beta$  is quite interesting. For  $\mu < 0$ , since the solutions  $\psi(x)$  are damped at  $\pm\infty$ , the density function for these states rapidly goes to zero as  $|x|$  increases. Thus these solutions may be interpreted as bound states in the continuum. In other words, the scattering states of the Hermitian problem get transformed to bound states in the continuum for the corresponding non-Hermitian problem. It is worth noting here that with modified definitions for current density and charge density for non-Hermitian generalized Swanson models, there is no violation in the equation of continuity in either case.

We have plotted the real part of the wave function  $\psi$  for the damped wave solution ( $\mu < 0$ ) for the Pöschl–Teller model in figure 1, while the wave function for a progressive solution ( $\mu > 0$ ) for the same model is plotted in figure 2. The Morse interaction shows similar behaviour.

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